# Teaching Natural Deduction as a Subversive Activity


James Caldwell

Department of Computer Science
University of Wyoming
Laramie, WY


Third International Congress on Tools for Teaching Logic
3 June 2011

# A Story

Lazy Professor: (To class) *Do exercise xxx from the text.*

## The Problematic Exercise

Prove the following formula using Natural Deduction.

$$((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \vee (Q \Rightarrow R))$$

... Two days pass.

# A Story

Lazy Professor: (To class) *Do exercise xxx from the text.*

## The Problematic Exercise

Prove the following formula using Natural Deduction.

$$((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \vee (Q \Rightarrow R))$$

... Two days pass.

A Chorus of Students: *There is a typo in the book!*

Student A: *The formula is false.*

## Student A: Disputation

If $P$ and $Q$ together obtain $R$, then surely it is not always the case that either $P$ alone or $Q$ alone obtains $R$ *e.g.* let $P$ be *over 18* and $Q$ be *male* and $R$ be *must register for military service*[a].

[a] In the US women do not register with the selective service.

# The story continues

Lazy Professor: *But it is valid.*
(Expressing sympathy and explaining that the counterexample, though compelling, is incorrectly formulated as stated; that predicates and quantifiers are required to formulate it and that in fact, when properly formulated, the obviously false thing is not valid.)
Student A: *Classical logic is clearly wrong.*
A Chorus of Students: *Yes, classical logic is wrong.*

... More time passes ...

# The story continues

Lazy Professor: *But it is valid.*
(Expressing sympathy and explaining that the counterexample, though compelling, is incorrectly formulated as stated; that predicates and quantifiers are required to formulate it and that in fact, when properly formulated, the obviously false thing is not valid.)
Student A: *Classical logic is clearly wrong.*
A Chorus of Students: *Yes, classical logic is wrong.*

... More time passes ...

Lazy Professor: *Consider the student's disputation of the following formula, is the student correct?*
Professor A: *It would seem so.*
Lazy Professor: *But the formula is valid*

After a moment of reflection ...

Professor A: *Ah, of course, there is no way to falsify it.*

# Some Remarks and a Claim

- $((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \vee (Q \Rightarrow R))$
  - A so-called "paradox" of material implication.
  - Generalization of De Morgan (take $R$ to be $\perp$ and $\neg\phi \stackrel{def}{=} \phi \Rightarrow \perp$) $\neg(P \wedge Q) \Rightarrow (\neg P \vee \neg Q)$
  - It is a superintuitionistic theorem of classical logic (not intuitionistically provable) – such theorems are difficult to prove in natural deduction.

# Some Remarks and a Claim

- $((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \vee (Q \Rightarrow R))$
  - A so-called "paradox" of material implication.
  - Generalization of De Morgan (take $R$ to be $\perp$ and $\neg\phi \stackrel{def}{=} \phi \Rightarrow \perp$)
    $\neg(P \wedge Q) \Rightarrow (\neg P \vee \neg Q)$
  - It is a superintuitionistic theorem of classical logic (not intuitionistically provable) – such theorems are difficult to prove in natural deduction.
- The Lazy Professor's students are in good company - Intuitionist and Relevant logicians reject the formula.

# Some Remarks and a Claim

- $((P \land Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \lor (Q \Rightarrow R))$
  - A so-called “paradox” of material implication.
  - Generalization of De Morgan (take $R$ to be $\bot$ and $\neg\phi \overset{def}{=} \phi \Rightarrow \bot$)
    $\neg(P \land Q) \Rightarrow (\neg P \lor \neg Q)$
  - It is a superintuitionistic theorem of classical logic (not intuitionistically provable) – such theorems are difficult to prove in natural deduction.
- The Lazy Professor's students are in good company - Intuitionist and Relevant logicians reject the formula.
- The failure to find a proof led to reflection on the meaning of the formula (this is good) and rejection of classical logic (is this bad?).

# Some Remarks and a Claim

- $((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \vee (Q \Rightarrow R))$
  - A so-called "paradox" of material implication.
  - Generalization of De Morgan (take $R$ to be $\bot$ and $\neg\phi \stackrel{def}{=} \phi \Rightarrow \bot$) $\neg(P \wedge Q) \Rightarrow (\neg P \vee \neg Q)$
  - It is a superintuitionistic theorem of classical logic (not intuitionistically provable) – such theorems are difficult to prove in natural deduction.
- The Lazy Professor's students are in good company - Intuitionist and Relevant logicians reject the formula.
- The failure to find a proof led to reflection on the meaning of the formula (this is good) and rejection of classical logic (is this bad?).

## A Claim

These difficulties do not arise when students are taught to do sequent proofs instead of Natural Deduction proofs.

# A Note on "Subversive" Activities

## Subversive?

This is what Neil Postman[a] called *Subversive Teaching* - the method leads students to raise interesting questions for themselves about accepted ideas.

[a] *Teaching as a Subversive Activity*, Neil Postman and Charles Weingartner, Delta Books, NY 1971.

# A Note on "Subversive" Activities

## Subversive?

This is what Neil Postman[a] called *Subversive Teaching* - the method leads students to raise interesting questions for themselves about accepted ideas.

[a] *Teaching as a Subversive Activity*, Neil Postman and Charles Weingartner, Delta Books, NY 1971.

## But is it the best way to teach classical logic?

Intuitionist and Relevant logicians raise serious questions about classical logic, but is a students first encounter with logic the best point to raise these questions?

# Gentzen's Proof Systems

# Natural Deduction and Sequent Proof Systems

Natural Deduction and Sequent proof systems were introduced by Gentzen in 1935 in his paper *Investigations into Logical Deduction*[1]

## Proof Systems

$\mathcal{NJ}$ Intuitionistic Natural Deduction

$\mathcal{NK}$ Classical Natural Deduction

$\mathcal{LJ}$ Intuitionistic Sequent Calculus

$\mathcal{LK}$ Classical Sequent Calculus

[1] M.E.Szabo, The Collected Works of Gerhard Gentzen, pp. 68-131, North Holland 1969.

# Natural Deduction and Sequent Proof Systems

Natural Deduction and Sequent proof systems were introduced by Gentzen in 1935 in his paper *Investigations into Logical Deduction*[1]

## Proof Systems

$\mathcal{NJ}$ Intuitionistic Natural Deduction

$\mathcal{NK}$ Classical Natural Deduction

$\mathcal{LJ}$ Intuitionistic Sequent Calculus

$\mathcal{LK}$ Classical Sequent Calculus

## Relationships

- $\mathcal{NK}$ is obtained from $\mathcal{NJ}$ by adding a rule for *Tertium non datur* or *Reductio ad Absurdum*.
- Rather surprisingly, $\mathcal{LK}$ is obtained from $\mathcal{LJ}$ simply by allowing multiple formula on the right side.

[1] M.E.Szabo, The Collected Works of Gerhard Gentzen, pp. 68-131, North Holland 1969.

# Natural Deduction Proof Rules

<u>Elimination Rules</u>

$$\frac{\phi \wedge \psi}{\phi}\wedge e_1 \qquad \frac{\phi \wedge \psi}{\psi}\wedge e_2$$

$$\frac{\phi \vee \psi \qquad \begin{array}{c} {}^{j} \\ [\phi] \\ \vdots \\ \sigma \end{array} \qquad \begin{array}{c} {}^{k} \\ [\psi] \\ \vdots \\ \sigma \end{array}}{\sigma}\vee e$$

$$\frac{\phi \qquad \phi \Rightarrow \psi}{\psi}\Rightarrow e$$

$$\frac{\phi \qquad \neg\phi}{\bot}\neg e$$

$$\frac{\bot}{\phi}\bot e$$

$$\frac{\begin{array}{c} {}^{j} \\ [\neg\phi] \\ \vdots \\ \bot \end{array}}{\phi}\text{RAA}$$

<u>Introduction Rules</u>

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}\wedge i$$

$$\frac{\phi}{\phi \vee \psi}\vee i_1 \qquad \frac{\psi}{\phi \vee \psi}\vee i_2$$

$$\frac{\begin{array}{c} {}^{j} \\ [\phi] \\ \vdots \\ \psi \end{array}}{\phi \Rightarrow \psi}\Rightarrow i$$

$$\frac{\begin{array}{c} {}^{j} \\ [\phi] \\ \vdots \\ \bot \end{array}}{\neg\phi}\neg i$$

# Comments on Natural Deduction

- There is an elegant symmetry in the Introduction and Elimination rules for the logical connectives ($\mathcal{NJ}$).

# Comments on Natural Deduction

- There is an elegant symmetry in the Introduction and Elimination rules for the logical connectives ($\mathcal{NJ}$).
- The Curry-Howard Isomorphism relates $\mathcal{NJ}$ proofs with lambda terms.
  - Proofs $\rightleftharpoons$ Programs
  - Propositions $\rightleftharpoons$ Types

# Comments on Natural Deduction

- There is an elegant symmetry in the Introduction and Elimination rules for the logical connectives ($\mathcal{NJ}$).
- The Curry-Howard Isomorphism relates $\mathcal{NJ}$ proofs with lambda terms.
  - Proofs $\rightleftharpoons$ Programs
  - Propositions $\rightleftharpoons$ Types
- $\mathcal{NK}$ proofs require a combination of forward and backward reasoning.

# Comments on Natural Deduction

- There is an elegant symmetry in the Introduction and Elimination rules for the logical connectives ($\mathcal{NJ}$).
- The Curry-Howard Isomorphism relates $\mathcal{NJ}$ proofs with lambda terms.
  - Proofs $\rightleftharpoons$ Programs
  - Propositions $\rightleftharpoons$ Types
- $\mathcal{NK}$ proofs require a combination of forward and backward reasoning.
- $\textsc{Raa}$
  - Any theorem of $\mathcal{NK}$ not provable in $\mathcal{NJ}$ will require a use of $\textsc{Raa}$ - such theorems are called superintuitionistic.
  - $\textsc{Raa}$ breaks the symmetry of the intro/elim rules.
  - There is a similarity between $\neg i$ and $\textsc{Raa}$ but the rule $\neg i$ is easily derived from $\Rightarrow i$ while $\textsc{Raa}$ is not derivable and introduces a negation.

# Comments on Natural Deduction

- There is an elegant symmetry in the Introduction and Elimination rules for the logical connectives ($\mathcal{NJ}$).
- The Curry-Howard Isomorphism relates $\mathcal{NJ}$ proofs with lambda terms.
  - Proofs $\rightleftharpoons$ Programs
  - Propositions $\rightleftharpoons$ Types
- $\mathcal{NK}$ proofs require a combination of forward and backward reasoning.
- $\textsc{Raa}$
  - Any theorem of $\mathcal{NK}$ not provable in $\mathcal{NJ}$ will require a use of $\textsc{Raa}$ - such theorems are called superintuitionistic.
  - $\textsc{Raa}$ breaks the symmetry of the intro/elim rules.
  - There is a similarity between $\neg i$ and $\textsc{Raa}$ but the rule $\neg i$ is easily derived from $\Rightarrow i$ while $\textsc{Raa}$ is not derivable and introduces a negation.
- $\mathcal{NK}$ does not enjoy the subformula property – proofs of superintuitionistic theorems containing no negations will will require the introduction of a negation.

# Comments on Natural Deduction

- There is an elegant symmetry in the Introduction and Elimination rules for the logical connectives ($\mathcal{NJ}$).
- The Curry-Howard Isomorphism relates $\mathcal{NJ}$ proofs with lambda terms.
  - Proofs $\rightleftharpoons$ Programs
  - Propositions $\rightleftharpoons$ Types
- $\mathcal{NK}$ proofs require a combination of forward and backward reasoning.
- $\textsc{Raa}$
  - Any theorem of $\mathcal{NK}$ not provable in $\mathcal{NJ}$ will require a use of $\textsc{Raa}$ - such theorems are called superintuitionistic.
  - $\textsc{Raa}$ breaks the symmetry of the intro/elim rules.
  - There is a similarity between $\neg i$ and $\textsc{Raa}$ but the rule $\neg i$ is easily derived from $\Rightarrow i$ while $\textsc{Raa}$ is not derivable and introduces a negation.
- $\mathcal{NK}$ does not enjoy the subformula property – proofs of superintuitionistic theorems containing no negations will will require the introduction of a negation.
- Failure to find a proof does not provide evidence one does not exist.

## Sequents

- A Sequent characterizes the state of a proof.

[2] Velleman, *How to Prove it: A Structured Approach*, Cambridge Press 2006

# Sequents

- A Sequent characterizes the state of a proof.
- A sequent is a pair of (possibly empty) formula lists $\langle \Gamma, \Delta \rangle$
  - We write $\Gamma \vdash \Delta$.
  - $\Gamma$ is the *antecedent*.
  - $\Delta$ is the *succedent*.
  - Velleman calls these *Givens* and *Goals*[2]

[2] Velleman, *How to Prove it: A Structured Approach*, Cambridge Press 2006

# Sequents

- A Sequent characterizes the state of a proof.
- A sequent is a pair of (possibly empty) formula lists $\langle \Gamma, \Delta \rangle$
  - We write $\Gamma \vdash \Delta$.
  - $\Gamma$ is the *antecedent*.
  - $\Delta$ is the *succedent*.
  - Velleman calls these *Givens* and *Goals*[2]
- The semantics of a sequent is given by:

$$[\![\Gamma \vdash \Delta]\!] \stackrel{def}{=} (\bigwedge_{\phi \in \Gamma} \phi) \Rightarrow \bigvee_{\psi \in \Delta} \psi$$

Thus, a sequent is valid if some formula on the left is false or all formulas on the left are true and some formula on the right is as well.

[2] Velleman, *How to Prove it: A Structured Approach*, Cambridge Press 2006

# Sequents

- A Sequent characterizes the state of a proof.
- A sequent is a pair of (possibly empty) formula lists $\langle \Gamma, \Delta \rangle$
  - We write $\Gamma \vdash \Delta$.
  - $\Gamma$ is the *antecedent*.
  - $\Delta$ is the *succedent*.
  - Velleman calls these *Givens* and *Goals*[2]
- The semantics of a sequent is given by:

$$[\![\Gamma \vdash \Delta]\!] \stackrel{def}{=} (\bigwedge_{\phi \in \Gamma} \phi) \Rightarrow \bigvee_{\psi \in \Delta} \psi$$

  Thus, a sequent is valid if some formula on the left is false or all formulas on the left are true and some formula on the right is as well.
- $\mathcal{LJ}$ restricts $|\Delta| \leq 1$ while $\mathcal{LK}$ has no restriction on the length of the succedent.

[2] Velleman, *How to Prove it: A Structured Approach*, Cambridge Press 2006

# Sequent Proof Rules

**AXIOMS**

$$\frac{}{\Gamma_1, \phi, \Gamma_2 \vdash \Delta_1, \phi, \Delta}(\mathsf{Ax}) \qquad \frac{}{\Gamma_1, \bot, \Gamma_2 \vdash \Delta}(\bot\mathsf{Ax})$$

**LEFT RULES**

$$\frac{\Gamma_1, \phi, \psi, \Gamma_2 \vdash \Delta}{\Gamma_1, \phi \wedge \psi, \Gamma_2 \vdash \Delta}(\wedge\mathsf{L})$$

$$\frac{\Gamma_1, \phi, \Gamma_2 \vdash \Delta \qquad \Gamma_1, \psi, \Gamma_2 \vdash \Delta}{\Gamma_1, \phi \vee \psi, \Gamma_2 \vdash \Delta}(\vee\mathsf{L})$$

$$\frac{\Gamma_1, \Gamma_2 \vdash \phi, \Delta \qquad \Gamma_1, \psi, \Gamma_2 \vdash \Delta}{\Gamma_1, \phi \Rightarrow \psi, \Gamma_2 \vdash \Delta}(\Rightarrow\mathsf{L})$$

$$\frac{\Gamma_1, \Gamma_2 \vdash \phi, \Delta}{\Gamma_1, \neg\phi, \Gamma_2 \vdash \Delta}(\neg\mathsf{L})$$

**RIGHT RULES**

$$\frac{\Gamma \vdash \Delta_1, \phi, \Delta_2 \qquad \Gamma \vdash \Delta_1, \psi, \Delta_2}{\Gamma \vdash \Delta_1, \phi \wedge \psi, \Delta_2}(\wedge\mathsf{R})$$

$$\frac{\Gamma \vdash \Delta_1, \phi, \psi, \Delta_2}{\Gamma, \vdash \Delta_1, \phi \vee \psi, \Delta_2}(\vee\mathsf{R})$$

$$\frac{\Gamma, \phi \vdash \Delta_1, \psi, \Delta_2}{\Gamma \vdash \Delta_1, \phi \Rightarrow \psi, \Delta_2}(\Rightarrow\mathsf{R})$$

$$\frac{\Gamma, \phi \vdash \Delta_1, \Delta_2}{\Gamma \vdash \Delta_1, \neg\phi, \Delta_2}(\neg\mathsf{R})$$

# Remarks on Sequent Rules and Derivations

- Left rules correspond to Elimination rules and Right rules correspond to Introduction rules - there is no rule corresponding to $\textsc{Raa}$.

# Remarks on Sequent Rules and Derivations

- Left rules correspond to Elimination rules and Right rules correspond to Introduction rules - there is no rule corresponding to $\textsc{Raa}$.
- Construction of sequent derivations is syntax driven.
  - Non-deterministically choose a compound formula in the left or right side and apply the corresponding rule.
  - If all formulas are atomic, check if the sequent is an instance of one of the axiom rules.
  - Repeat until all leaves of the tree are instances of axioms (and you have a proof) or until some atomic sequent turns out *not* to be an instance of an axiom rule (and you can build a counter example.)

# Remarks on Sequent Rules and Derivations

- Left rules correspond to Elimination rules and Right rules correspond to Introduction rules - there is no rule corresponding to $\textsc{Raa}$.
- Construction of sequent derivations is syntax driven.
  - Non-deterministically choose a compound formula in the left or right side and apply the corresponding rule.
  - If all formulas are atomic, check if the sequent is an instance of one of the axiom rules.
  - Repeat until all leaves of the tree are instances of axioms (and you have a proof) or until some atomic sequent turns out *not* to be an instance of an axiom rule (and you can build a counter example.)
- Failed derivations yield counterexamples.
  - Consider an atomic sequent of the form $\Gamma \vdash \Delta$ that is *not* an instance of an axiom rule
  - The assignment $(\lambda x.\textit{if}\,(x \in \Delta)$ *then* True *else* False) falsifies $\Gamma \vdash \Delta$
  - It also falsifies any sequent rooting a derivation ending with $\Gamma \vdash \Delta$.
- Proofs from assumptions are obtained by adding the assumed formulas to $\Gamma$.

# $\mathcal{NK}$ Proof of Excluded Middle

- We will prove $\phi \vee \neg\phi$ in both $\mathcal{NK}$ and $\mathcal{LK}$.
- Examining the rules, it becomes clear that the explicit intro rules $\vee i_1$ and $\vee i_2$ can only arise from a proof of $\phi$ or a proof of $\neg\phi$ which is impossible without assumptions.
- The only rule that can help is $\textsc{Raa}$.

$$\dfrac{\dfrac{\overset{\cancel{1}}{[\neg(\phi \vee \neg\phi)]}}{\begin{array}{c}\vdots\\ \bot\end{array}}}{\phi \vee \neg\phi}\;\textsc{Raa}\;[1]$$

- We must derive $\bot$ from the assumption $\neg(\phi \vee \neg\phi)$.

# $\mathcal{NK}$ Proof of Excluded Middle (Cont.)

- The only rule having $\perp$ as its conclusion is $\neg e$.

$$
\dfrac{\dfrac{\dfrac{\overset{\not{1}}{[\neg(\phi \vee \neg\phi)]}}{}\quad}{\begin{array}{c}\vdots\\ \phi \vee \neg\phi\end{array}} \qquad \overset{\not{1}}{[\neg(\phi \vee \neg\phi)]}}{{}^{[1]}\dfrac{\perp}{\phi \vee \neg\phi}\,\mathrm{RAA}}\,\neg e
$$

- Now we must derive $\phi \vee \neg\phi$ from $\neg(\phi \vee \neg\phi)$.

# $\mathcal{NK}$ Proof of Excluded Middle (Cont.)

- Again, $\neg e$ may be able to help. We need $\phi \vee \neg\phi$ so we assume $\phi$ and then use $\vee i_1$.

$$\cfrac{\cfrac{\cfrac{\overset{2}{[\phi]}}{\phi \vee \neg\phi}\,\vee i_1 \qquad \overset{\cancel{1}}{[\neg(\phi \vee \neg\phi)]}}{\cfrac{\bot}{\begin{array}{c}\vdots\\ \phi \vee \neg\phi\end{array}}}\,\neg e \qquad\qquad \overset{\cancel{1}}{[\neg(\phi \vee \neg\phi)]}}{{}^{[1]}\,\cfrac{\bot}{\phi \vee \neg\phi}\,\mathrm{RAA}}\,\neg e$$

- Now of course we have a new hypothesis to discharge.

# $\mathcal{NK}$ Proof of Excluded Middle

- An application of $\neg i$ can be used to discharge hypothesis 2 and yields $\neg\phi$.

$$
{\scriptstyle [1]}\,\dfrac{\dfrac{\dfrac{{\scriptstyle [2]}\,\dfrac{\dfrac{\dfrac{\overset{\cancel{2}}{[\phi]}}{\phi \lor \neg\phi}\,{\lor}i_1 \qquad \overset{\cancel{1}}{[\neg(\phi \lor \neg\phi)]}}{\bot}\,\neg e}{\neg\phi}\,\neg i}{\begin{array}{c}\vdots\\ \phi \lor \neg\phi\end{array}} \qquad \overset{\cancel{1}}{[\neg(\phi \lor \neg\phi)]}}{\bot}\,\neg e}{\phi \lor \neg\phi}\,\mathrm{RAA}
$$

- Now the goal is to derive $\phi \lor \neg\phi$ from $\neg\phi$. This is easily done with the rule ${\lor}i_2$.

# $\mathcal{NK}$ proof of Excluded Middle (Cont.)

- This completes the proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\cancel{2}}{[\phi]}}{\phi \vee \neg\phi}\,\vee i_1 \qquad \overset{\cancel{1}}{[\neg(\phi \vee \neg\phi)]}}{\bot}\,\neg e}{[2]\;\neg\phi}\,\neg i}{\phi \vee \neg\phi}\,\vee i_2 \qquad \overset{\cancel{1}}{[\neg(\phi \vee \neg\phi)]}}{\bot}\,\neg e}{[1]\;\phi \vee \neg\phi}\,\mathrm{R{\small AA}}$$

- This is the shortest natural deduction proof of excluded middle the author knows of.

# $\mathcal{LK}$ proof of Excluded Middle

- Now we derive the sequent $\vdash \phi \vee \neg\phi$ having no assumptions.

# $\mathcal{LK}$ proof of Excluded Middle

- Now we derive the sequent $\vdash \phi \vee \neg\phi$ having no assumptions.
- The only rule that applies here is $\vee R$.

$$\frac{\vdash \phi, \neg\phi}{\vdash \phi \vee \neg\phi}\ \vee r$$

# $\mathcal{LK}$ proof of Excluded Middle

- Now we derive the sequent $\vdash \phi \vee \neg\phi$ having no assumptions.
- The only rule that applies here is $\vee R$.

$$\dfrac{\vdash \phi, \neg\phi}{\vdash \phi \vee \neg\phi}\ \vee r$$

- At this point we have two goals, it is enough to prove either one.
- The only rule that applies is $\neg R$.

$$\dfrac{\dfrac{\phi \vdash \phi}{\vdash \phi, \neg\phi}\ \neg R}{\vdash \phi \vee \neg\phi}\ \vee R$$

# $\mathcal{LK}$ proof of Excluded Middle

- Now we derive the sequent $\vdash \phi \vee \neg\phi$ having no assumptions.
- The only rule that applies here is $\vee R$.

$$\frac{\vdash \phi, \neg\phi}{\vdash \phi \vee \neg\phi}\vee r$$

- At this point we have two goals, it is enough to prove either one.
- The only rule that applies is $\neg R$.

$$\cfrac{\cfrac{\phi \vdash \phi}{\vdash \phi, \neg\phi}\neg R}{\vdash \phi \vee \neg\phi}\vee R$$

- This is an instance of the axiom rule and the proof is complete.

$$\cfrac{\cfrac{\cfrac{}{\phi \vdash \phi}\text{Ax}}{\vdash \phi, \neg\phi}\neg R}{\vdash \phi \vee \neg\phi}\vee R$$

# $\mathcal{NK}$ derivation of GDM

$$
\text{[1]}\;\dfrac{\text{[2]}\;\dfrac{\dfrac{\dfrac{\text{[4]}\;\dfrac{\dfrac{\dfrac{\dfrac{\text{[3]}\;\dfrac{\dfrac{\dfrac{\overset{\cancel{3}}{[\phi]}\quad\overset{\cancel{4}}{[\psi]}}{\phi\wedge\psi}\wedge i\quad\overset{\cancel{1}}{[(\phi\wedge\psi)\Rightarrow\sigma\,]}}{\sigma}\Rightarrow e}{\phi\Rightarrow\sigma}\Rightarrow i}{(\phi\Rightarrow\sigma)\vee(\psi\Rightarrow\sigma)}\vee i_1\quad\overset{\cancel{2}}{[\neg((\phi\Rightarrow\sigma)\vee(\psi\Rightarrow\sigma))]}}{\bot}\neg e}{\sigma}\bot e}{\psi\Rightarrow\sigma}\Rightarrow i}{(\phi\Rightarrow\sigma)\vee(\psi\Rightarrow\sigma)}\vee i_2\quad\overset{\cancel{2}}{[\neg((\phi\Rightarrow\sigma)\vee(\psi\Rightarrow\sigma))]}}{\bot}\neg e}{(\phi\Rightarrow\sigma)\vee(\psi\Rightarrow\sigma)}\,\mathrm{RAA}}{((\phi\wedge\psi)\Rightarrow\sigma)\Rightarrow(\phi\Rightarrow\sigma)\vee(\psi\Rightarrow\sigma)}\Rightarrow i
$$

# Sequent derivation of GDM

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\phi, \psi \vdash \phi, \sigma$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\phi, \psi \vdash \psi, \sigma$}
\RightLabel{$\wedge$R}
\BinaryInfC{$\phi, \psi \vdash \phi \wedge \psi, \sigma$}
\RightLabel{$\Rightarrow$R}
\UnaryInfC{$\phi \vdash \phi \wedge \psi, \psi \Rightarrow \sigma$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\sigma, \phi \vdash \sigma, \psi \Rightarrow \sigma$}
\RightLabel{$\Rightarrow L$}
\BinaryInfC{$(\phi \wedge \psi) \Rightarrow \sigma, \phi \vdash \sigma, \sigma \Rightarrow \psi$}
\RightLabel{$\Rightarrow$R}
\UnaryInfC{$(\phi \wedge \psi) \Rightarrow \sigma \vdash \phi \Rightarrow \sigma, \psi \Rightarrow \sigma$}
\RightLabel{$\vee$R}
\UnaryInfC{$(\phi \wedge \psi) \Rightarrow \sigma \vdash (\phi \Rightarrow \sigma) \vee (\psi \Rightarrow \sigma)$}
\RightLabel{$\Rightarrow$R}
\UnaryInfC{$\vdash ((\phi \wedge \psi) \Rightarrow \sigma) \Rightarrow ((\phi \Rightarrow \sigma) \vee (\psi \Rightarrow \sigma))$}
\end{prooftree}
$$

# Against Sequents

- It is often claimed that natural deduction proofs reflect the mathematical thought process but that sequents do not.
  - I claim it is the rules, not the tree structure of a proof, that serves to "explain" the logical laws.
  - Consider disjunction: "To prove $\phi \vee \psi$ prove $\psi$ or prove $\psi$."

$$\text{Natural Deduction} \qquad\qquad \text{Sequent Calculus}$$

$$\frac{\phi}{\phi \vee \psi}\vee i_1 \quad \frac{\psi}{\phi \vee \psi}\vee i_2 \qquad\qquad \frac{\Gamma \vdash \Delta_1, \phi, \psi, \Delta_2}{\Gamma, \vdash \Delta_1, \phi \vee \psi, \Delta_2}(\vee R)$$

  - The sequent rule works perfectly well as an explanation.

# Against Sequents

- It is often claimed that natural deduction proofs reflect the mathematical thought process but that sequents do not.
  - I claim it is the rules, not the tree structure of a proof, that serves to "explain" the logical laws.
  - Consider disjunction: "To prove $\phi \vee \psi$ prove $\psi$ or prove $\psi$."

$$\begin{array}{cc} \text{Natural Deduction} & \text{Sequent Calculus} \\ \dfrac{\phi}{\phi \vee \psi}\vee i_1 \quad \dfrac{\psi}{\phi \vee \psi}\vee i_2 & \dfrac{\Gamma \vdash \Delta_1, \phi, \psi, \Delta_2}{\Gamma, \vdash \Delta_1, \phi \vee \psi, \Delta_2}(\vee R) \end{array}$$

  - The sequent rule works perfectly well as an explanation.
- Sequent proofs require too much writing.
  - Not in superintuitionistic cases (see examples above).
  - Ink is cheap.
  - With tool support this claim is moot.
  - It is perfectly OK to elide unneeded formulas in the antecedent.

$$\dfrac{\Gamma' \vdash \Delta}{\Gamma, \vdash \Delta}(\text{Thin}) \text{ where } \Gamma' \subseteq \Gamma$$

# Against Sequents

- Multiple formulas on the right are hard to motivate.
  - The sequent semantics and $\vee R$ are justification enough.

$$[\![\Gamma \vdash \Delta]\!] \stackrel{def}{=} (\bigwedge_{\phi \in \Gamma} \phi) \Rightarrow \bigvee_{\psi \in \Delta} \psi$$

  - Velleman (a best selling book on informal proof methods) uses them.

# Against Sequents

- Multiple formulas on the right are hard to motivate.
  - The sequent semantics and $\vee R$ are justification enough.

$$[\![\Gamma \vdash \Delta]\!] \stackrel{def}{=} (\bigwedge_{\phi \in \Gamma} \phi) \Rightarrow \bigvee_{\psi \in \Delta} \psi$$

  - Velleman (a best selling book on informal proof methods) uses them.
- Cut is hard to motivate?
  - I find this one hard to understand.

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma, \phi \vdash \Delta}{\Gamma \vdash \Delta}(\text{Cut})$$

  - Cut is not needed though it can be convenient in predicate logic proofs.
  - More conveniently, we add a lemma rule for previously proved theorems.

$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \Delta}(\text{Lemma}) \text{ where } \vdash \phi$$

# For Sequents

- The provide a decision procedure for propositional logic.
- Counter-examples are easily generated from failed proofs.
- Students gain confidence as they become more adept at manipulating the formalism.
- There is no question "Is this a proof?"
- Curry-Howard still holds (for $\mathcal{LJ}$ proofs.)
- It is easy to identify the superintuitionistic theorems – which ones necessarily have two formulas on the right at some point in the derivation.

# Propositional Logic is Easy!

**Teaching propositional logic using Natural Deduction is a bit like teaching arithmetic using Roman numerals**

You could force students to suffer through it, but aren't Arabic numerals better suited to the task?